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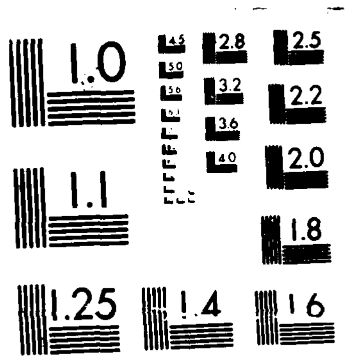
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INFLUENCE OF ELECTROMAGNETIC EFFECTS ON THE TWO STREAM INSTABILITY IN A RELATIVISTIC ELECTRON BEAM

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INFLUENCE OF ELECTROMAGNETIC EFFECTS ON THE TWO STREAM INSTABILITY IN A RELATIVISTIC ELECTRON BEAM

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This paper investigates the influence of electromagnetic effects on the two stream instability in a relativistic electron beam propagating through a collisionless plasma channel. The analysis is carried out within the framework of a macroscopic cold fluid model in which beam and plasma fluid element is in a laminar flow. Axisymmetric electromagnetic stability properties are calculated for the case in which the equilibrium beam and plasma density profiles are rectangular. Consistent with the two stream instability, the perturbed fields are assumed to be the transverse magnetic mode. The resulting eigenvalue equation for the perturbed axial electric field $\delta\hat{E}_z$ is solved to give a closed algebraic dispersion relation for the complex eigenfrequency ω. This dispersion relation is solved and it is shown that the electromagnetic effects have a strong stabilizing influence for a relativistic electron beam with $\gamma_b >> 1$, where γ_b is the relativistic mass ratio of beam electrons. For example, the critical beam current for instability is proportional to the electromagnetic current enhancement factor $\xi = (\gamma_b + 1)/2$. Thus, the critical current from the electromagnetic calculation increases more drastically with the beam energy than that from the electrostatic approximation.

I. INTRODUCTION

In recent years, there have been intense research activities of the equilibrium[1,2] and stability[3-6] properties of relativistic electron beam propagating through a background plasma channel or a gaseous medium. Perhaps one of the most basic instabilities that characterizes a relativistic electron beam propagating through a collisionless plasma is the two stream instability,[6-9] which results from the relative drift motion between the beam electrons and the background plasma particles. Although the two stream instability is very familiar in the plasma physics community, most of the previous studies[7-8] on this instability have been limited to one dimensional calculations. In recent literature,[6] Bogdankevich and Rukhadze investigated the two stream instability of a relativistic electron beam, including the finite radial geometry effects on stability behavior. Therefore, they were able to determine the limiting beam current due to the two stream instability. In addition, we recently developed a theory of the two stream instability in a self-pinched relativistic electron beam,[9] finding out the critical beam current for instability. However, both previous calculations[6,9] have been based on the electrostatic approximation. Although this is a reasonable approximation for a mildly relativistic electron beam, we expect significant modifications to the stability behavior when the beam energy is ultrarelativistic. For a relativistic beam, the electromagnetic effects on the beam dynamics are often very important. In this regard, in this paper, we investigate the influence of electromagnetic effects on the two stream instability in a relativistic electron beam propagating through a collisionless plasma channel.

The analysis in this article is carried out within the framework of a macroscopic cold fluid model, assuming either that the beam-plasma fluids are immersed in a uniform axial magnetic field or that the beam is self-pinched. Theoretical model and the equilibrium properties of the beam-plasma configuration will be briefly discussed in Sec. II for rectangular density profiles of the beam and plasma particles. The stability analysis presented in Sec. III assume axisymmetric electromagnetic perturbations $(\partial/\partial\theta = 0)$. Moreover, since the unstable mechanism of the two stream instability is mostly due to fluctuations of the axial electron field, the stability analysis in this article is restricted to perturbations of the transverse magnetic (TM) mode polarization. The dispersion relation of the axisymmetric transverse magnetic mode is derived in Sec. III for rectangular density profiles of the beam and plasma particles.

Assuming long axial wavelength perturbations in Sec. IV, we apply this dispersion relation to two specific cases: (a) highly magnetized bounded plasma, and (b) a unmagnetized self-pinched electron beam. In case (a), the dominant two stream instability results from the relative drift motion between the beam electrons and plasma electrons. One of the most important features of the analysis in Sec. IV is that the critical beam current for instability is proportional to the electromagnetic current enhancement factor. After some analytical algebraic manipulation, this current enhancement factor ξ_{th} is approximately given by [Eq. (67)]

$$\xi_{th} = \frac{\gamma_b + 1}{2}$$

for the equidensity case where the beam and plasma electrons have the same number density. Here $\gamma_b mc^2$ is the characteristic beam electron energy, m is the electron rest mass and c is the speed of light in vacuum. In the limiting case when $\gamma_b = 1$, this enhancement factor recovers the electrostatic approximation $\xi_{th} = 1$. Obviously, the critical beam current for instability increases drastically with the beam energy (γ_b), clearly demonstrating that the electromagnetic effects play a major role in the two stream stability behavior for a relativistic beam with $\gamma_b >> 1$. In Sec. IV, a numerical investigation of the dispersion relation is also carried out and it is shown that the analytical expression of the current enhancement factor in Eq. (67) underestimates the true electromagnetic current enhancement. Finally, two stream stability properties in a self-pinched electron beam are also briefly discussed in Sec. IV.

II. THEORETICAL MODEL AND EQUILIBRIUM PROPERTIES

II.A Theoretical Model

In the present analysis, the beam and plasma particles are treated as a cold fluid immersed in a uniform axial magnetic field $B_0\hat{\mathbf{e}}_z$. In the limiting case when the applied axial magnetic field vanishes, the beam electrons are in a self-pinched equilibrium where the self magnetic force counterbalances the centrifugal force resulted from the beam rotation. Within the context of the macroscopic cold fluid description, the equation of motion and the continuity equation for the beam and plasma fluid element can be expressed as

$$\frac{\partial}{\partial t}\mathbf{P}_j + \mathbf{V}_j \cdot \nabla\mathbf{P}_j = e_j\left(\mathbf{E} + \frac{\mathbf{V}_j \times \mathbf{B}}{c}\right), \tag{1}$$

$$\frac{\partial}{\partial t} n_j + \nabla \cdot \left(n_j\mathbf{V}_j\right) = 0, \tag{2}$$

where the subscript j = b, i, e denotes beams, ions, plasma electrons, respectively, $\mathbf{P}_j$ is the mean mechanical momentum, $\mathbf{V}_j$ is the mean velocity, e_j is the charge, and n_j is the density. In Eq. (1), $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ are the electric and magnetic fields which are self-consistently determined from the Maxwell equations, that is,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}, \tag{3}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\sum_j e_j n_j \mathbf{V}_j + \frac{4\pi}{c}\mathbf{J}_{ext} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}, \tag{4}$$

$$\nabla \cdot \mathbf{E} = \sum_j 4\pi n_j e_j + 4\pi\rho_{ext}, \tag{5}$$

where ρ_{ext} and $\underset{\sim}{J}_{ext}$ are externally provided charge and current densities.

II.B General Equilibrium Properties

An equilibrium analysis of Eqs. (1) - (5) is carried out by setting $\partial/\partial t = 0$. Thus, the equilibrium properties are readily obtained from

$$\nabla \cdot \left(n_j^0 \underset{\sim}{V}_j^0\right) = 0, \tag{6}$$

$$\underset{\sim}{V}_j^0 \cdot \nabla \underset{\sim}{P}_j^0 = e_j \left(\underset{\sim}{E}^0 + \frac{\underset{\sim}{V}_j^0 \times \underset{\sim}{B}^0}{c}\right), \tag{7}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B}^0 = \frac{4\pi}{c} \sum_j e_j n_j^0 \underset{\sim}{V}_j^0 + \frac{4\pi}{c} \underset{\sim}{J}_{ext}(\underset{\sim}{x}), \tag{8}$$

$$\nabla \cdot \underset{\sim}{E}^0 = \sum_j 4\pi n_j^0 e_j + 4\pi \rho_{ext}(\underset{\sim}{x}), \tag{9}$$

where $n_j^0(\underset{\sim}{x})$, $\underset{\sim}{V}_j^0(\underset{\sim}{x})$, $\underset{\sim}{P}^0(\underset{\sim}{x})$, $\underset{\sim}{E}^0(\underset{\sim}{x})$ and $\underset{\sim}{B}^0(\underset{\sim}{x})$ are the macroscopic equilibrium quantities.

We introduce cylindrical polar coordinates (r, θ, z) with z-axis corresponding to the axis of symmetry; r is the radial distance from the z-axis, and θ is the polar angle in a plane perpendicular to the z-axis. For azimuthally symmetric particle equilibrium ($\partial/\partial\theta = 0$ and $\partial/\partial z = 0$) characterized by $n_j^0(r)$ and $\underset{\sim}{V}_j^0 = V_{j\theta}^0(r)\hat{\underset{\sim}{e}}_\theta + V_{jz}^0(r)\hat{\underset{\sim}{e}}_z$, it is straightforward to show from Eq. (6) that the functional form of particle density profile $n_j^0(r)$ can be specified arbitrarily. Moreover, the deviation from equilibrium charge and

current neutrality produces a radial electric and azimuthal magnetic field that influences the azimuthal motion of particle fluid. It follows from Eq. (7) that equilibrium force balance in the radial direction can be expressed as

$$-m_j \frac{V_{j\theta}^{02}(r)}{r} = e_j\left[E_r^0(r) - \frac{V_{jz}^0}{c} B_\theta^0(r) + \frac{V_{j\theta}^0}{c} B_0\right], \tag{10}$$

where m_j for beam and plasma electrons are expressed as $m_b = \gamma_b m$ and $m_e = m$, respectively, and the equilibrium axial velocity profile of the beam electrons is independent of r, i.e., $V_{bz}^0(r) = \beta_b c = \text{const.}$, which relates to γ_b by $\gamma_b^2 = \left(1 - \beta_b^2\right)^{-1}$. We also assume that the axial velocity of ions and plasma electrons is zero, although this restriction can be easily eliminated. However, this restriction does not effect on the electromagnetic influence of the two stream instability. In Eq. (10), we neglect the self axial magnetic field, assuming that the azimuthal motion of particle fluid element is nonrelativistic.

With $\rho_{ext} = 0$, the equilibrium radial electric field is determined from Eq. (9) and is given by

$$E_r^0(r) = \frac{4\pi}{r} \sum_j e_j \int_0^r dr' r' n_j^0(r'). \tag{11}$$

Similarly, it can be shown that the equilibrium magnetic field is expressed as

$$B_\theta^0(r) = -4\pi\beta_b e \frac{1}{r} \int_0^r dr' r' n_b^0(r'). \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (10), we can obtain the radial force equation explicitly.

II.C Sharp-Boundary Equilibrium

Although the formalism outlined in Sec. II.B can be useful to investigate equilibrium properties for a broad class of density profiles, for purposes of analytic simplification in the stability analysis, we specialize to the case of a sharp-boundary equilibrium in which the particle density profiles can be expressed as

$$n_j^0(r) = \begin{cases} \hat{n}_j = \text{const.}, & 0 < r < R_b, \\ 0, & R_b < r < R_c. \end{cases} \tag{13}$$

In Eq. (13), R_b denotes the radius of the beam and plasma column. As a general description, we also assume that the beam-plasma system is bounded by an outer conductor with radius R_c. However, in a special occasion when $R_c \to \infty$, the system becomes unbounded. For convenience in the subsequent analysis, we introduce

$$f_i = \hat{n}_i/\hat{n}_b, \; f_e = \hat{n}_e/\hat{n}_b, \tag{14}$$

where f_i and f_e are positive constants.

The equilibrium radial electric field in Eq. (11) is expressed as

$$E_r^0(r) = 2\pi e\hat{n}_b(f_i - f_e - 1) \cdot \begin{cases} r, & 0 < r < R_b, \\ R_b^2/r, & R_b < r < R_c. \end{cases} \tag{15}$$

Substituting Eq. (13) into Eq. (12) gives the azimuthal magnetic field

$$B_\theta^0(r) = -2\pi\beta_b e\hat{n}_b \begin{cases} r, & 0 < r < R_b, \\ R_b^2/r, & R_b < r < R_c. \end{cases} \qquad (16)$$

Laminar rotational frequencies of the beam and plasma fluid elements are determined from Eq. (10) by substituting Eqs. (15) and (16) into Eq. (10) and carrying out a straightforward algebra. Defining the laminar rotational frequency of the beam electron by $\omega_b = V_{b\theta}^0(r)/r$, we obtain

$$\omega_b = \omega_b^\pm = \frac{\omega_{cb}}{2} \pm \left[\frac{\omega_{cb}^2}{4} - \frac{\omega_{pb}^2}{2}\left(\frac{1}{\gamma_b^2} + f_e - f_i\right)\right]^{1/2}, \qquad (17)$$

where $\omega_{cb} = eB_0/\gamma_b mc$ is the beam electron cyclotron frequency, $\omega_{pb}^2 = 4\pi e^2\hat{n}_b/\gamma_b m$ is the beam plasma frequency-squared and $\gamma_b^2 = (1 - \beta_b^2)^{-1}$. In Eq. (17), the upper sign $(\omega_b = \omega_b^+)$ corresponds to a "fast" rotational equilibrium, and the lower sign $(\omega_b = \omega_b^-)$ corresponds to a "slow" rotational equilibrium.

Similarly, we also obtain the rotational frequency of the plasma ions and electrons

$$\omega_i = \omega_i^\pm = -\frac{\omega_{ci}}{2} \mp \left[\frac{\omega_{ci}^2}{4} + \frac{1}{2}\eta\omega_{pb}^2\,(1 + f_e - f_i)\right]^{1/2}, \qquad (18)$$

$$\omega_e = \omega_e^\pm = \frac{\omega_{ce}}{2} \pm \left[\frac{\omega_{ce}^2}{4} - \frac{1}{2}\gamma_b\omega_{pb}^2\,(1 + f_e - f_i)\right]^{1/2}, \qquad (19)$$

respectively. In Eqs. (18) and (19), ω_{ci} and ω_{ce} are the cyclotron frequencies of the plasma ions and electrons, respectively, and $\eta = \gamma_b m/m_i$. From Eqs. (17) - (19), we require

$$\omega_{pb}^2 \left(1 + f_e - f_i\right) < \omega_{pb}^2 \beta_b^2 + \frac{1}{2} \omega_{cb}^2, \tag{20}$$

and

$$-\omega_{ci}^2/2 < \omega_{pb}^2 \left(1 + f_e - f_i\right) < \omega_{ce}^2/2\gamma_b, \tag{21}$$

for radial confinement of the equilibrium. The inequalities in Eqs. (20) and (21) assures that the repulsive space-charge force on the beam and plasma fluid element is weaker than the magnetic focussing force.

III. ELECTROMAGNETIC STABILITY ANALYSIS

In this section, we linearize Eqs. (1) - (5) assuming electromagnetic perturbations about the axisymmetric equilibria described in Sec. II. As indicated in the introduction, the present analysis is concentrated on the two stream instability which has dominant electric field polarization along the z-axis. In this regard, we assume that all perturbations have spatial dependence only on the z coordinate, according to

$$\delta\psi(\underset{\sim}{x},t) = \delta\hat{\psi}(r)\exp\{i(kz - \omega t)\}, \tag{22}$$

where $\delta\psi(\underset{\sim}{x},t)$ represents a perturbed quantity, k is the axial wavenumber and ω is the complex eigenfrequency. The assumption of the axisymmetric perturbation in Eq. (22) simplifies the subsequent stability analysis considerably. In addition, the stability analysis in this article is restricted to the transverse magnetic mode polarization consistent with the electric field polarization of the two stream instability. Therefore, the perturbed field equations linearized from Eqs. (3) and (4) are expressed as

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{\omega^2}{c^2} - k^2\right)\delta\hat{E}_z(r) = 4\pi ik\left[\delta\hat{\rho}(r) - \frac{\omega}{c^2k}\delta\hat{J}_z(r)\right], \tag{23}$$

$$ik\delta\hat{E}_r(r) - \frac{d}{dr}\delta\hat{E}_z(r) = i\frac{\omega}{c}\delta\hat{B}_\theta(r), \tag{24}$$

and

$$-ik\delta\hat{B}_\theta(r) = \frac{4\pi}{c}\delta\hat{J}_r(r) - i\frac{\omega}{c}\delta\hat{E}_r(r), \tag{25}$$

where $\delta\hat{\underset{\sim}{E}}$ and $\delta\hat{\underset{\sim}{B}}$ are the perturbed electric and magnetic fields, respectively, and $\delta\hat{\rho}(r)$ and $\delta\hat{\underset{\sim}{J}}(r)$ are perturbed charge and current densities, which are explicitly defined by

$$\delta\hat{\rho}(r) = \sum_j e_j \delta n_j, \tag{26}$$

$$\delta\hat{\underset{\sim}{J}}(r) = \sum_j e_j (\underset{\sim}{V}_j^0 \delta n_j + n_j^0 \delta \underset{\sim}{V}_j), \tag{27}$$

respectively.

The linearized equation of motion and the linearized continuity equation for the beam and plasma fluid element are expressed as

$$\frac{\partial}{\partial t} \delta \underset{\sim}{P}_j + \underset{\sim}{V}_j^0 \cdot \nabla \delta \underset{\sim}{P}_j + \delta \underset{\sim}{V}_j \cdot \nabla \underset{\sim}{P}_j^0 = e_j \left(\delta \underset{\sim}{E} + \frac{\underset{\sim}{V}_j^0 \times \delta \underset{\sim}{B}}{c} + \frac{\delta \underset{\sim}{V}_j \times \underset{\sim}{B}^0}{c} \right), \tag{28}$$

and

$$\frac{\partial}{\partial t} \delta n_j + \nabla \cdot \left(n_j^0 \delta \underset{\sim}{V}_j + \delta n_j \underset{\sim}{V}_j^0 \right) = 0, \tag{29}$$

from Eqs. (1) and (2). Before proceeding further stability analysis, we restrict the subsequent stability study to the following cases: (a) in the presence of an applied axial magnetic field $(B_0 \neq 0)$, the beam density is limited to satisfy

$$\frac{\nu}{3} \frac{\omega_{pb}^2}{\omega_{cb}^2} \beta_b^2 < < 1, \tag{30}$$

where ν is Budker's parameter of the beam defined by $\nu = e^2 N_b / mc^2$, and N_b is

the number of beam electrons per unit axial length, or (b) in case $B_0 = 0$, self-pinched electron beam propagating through a pure ion channel with $f_i < 1$ where the plasma electrons are not allowed $(f_e = 0)$ in order to satisfy the equilibrium condition in Eq. (21). Within the context of this restriction, we neglect the terms proportional to $B_\theta^0(r)$ in Eq. (28), substantially simplifying the subsequent stability analysis.

After a straightforward algebra with Eqs. (28) and (29), the linearized fluid calculation gives

$$-i(\omega - kV_{jz}^0)\ \delta\hat{V}_{jr} - (\epsilon_j\omega_{cj} + 2\omega_j)\ \delta\hat{V}_{j\theta} = \frac{e_j}{m_j}(\delta\hat{E}_r - V_{jz}^0\delta\hat{B}_\theta/c), \quad (31)$$

$$-i(\omega - kV_{jz}^0)\ \delta\hat{V}_{j\theta} + (\epsilon_j\omega_{cj} + 2\omega_j)\ \delta\hat{V}_{jr} = 0, \quad (32)$$

$$-i(\omega - kV_{jz}^0)\ \delta\hat{V}_{jz} = \frac{e_j}{\gamma_j^2 m_j}\delta\hat{E}_z, \quad (33)$$

$$-i(\omega - kV_{jz}^0)\ \delta\hat{n}_j + \frac{1}{r}\frac{\partial}{\partial r}(rn_j^0\delta\hat{V}_{jr}) + ikn_j^0\delta\hat{V}_{jz} = 0, \quad (34)$$

where the relativistic mass ratio γ_j for ions and plasma electrons is unity, $m_b = \gamma_b m$ and $\epsilon_j = \mathrm{sgn}\,e_j$.

From Eqs. (24) and (25), it is straightforward to show that the combination $\delta\hat{E}_r - V_{jz}^0\delta\hat{B}_\theta/c$ in the right-hand side of Eq. (31) is expressed as

$$\delta\hat{E}_r - \frac{V^0_{jz}}{c}\delta\hat{B}_\theta = -i\left(1 - \frac{\omega}{kc^2}V^0_{jz}\right)\frac{k}{p^2}\frac{d}{dr}\delta\hat{E}_z$$

$$+ \frac{4\pi i}{p^2c^2}\left(\omega - kV^0_{jz}\right)\sum_j e_j n^0_j \delta\hat{V}_{jr}, \tag{35}$$

where the parameter p is defined by

$$p^2 = k^2 - \omega^2/c^2. \tag{36}$$

After a tedious algebraic manipulation which makes use of Eqs. (31), (32) and (35), we obtain the coupled equations

$$\left[\left(\frac{\omega^2_{pb}}{p^2c^2} + 1\right)\left(\omega - k\beta_b c\right)^2 - \left(\omega_{cb} - 2\omega_b\right)^2\right]\delta\hat{V}_{br}$$

$$+ \frac{\omega^2_{pb}}{p^2c^2}\left(\omega - k\beta_b c\right)^2\left(f_e\delta\hat{V}_{er} - f_i\delta\hat{V}_{ir}\right)$$

$$= -\frac{e}{\gamma_b m}\left(\omega - k\beta_b c\right)\left(1 - \frac{\omega\beta_b}{kc}\right)\frac{k}{p^2}\frac{d}{dr}\delta\hat{E}_z, \tag{37}$$

$$\left[\left(\gamma_b f_e\frac{\omega^2_{pb}}{p^2c^2} + 1\right)\omega^2 - \left(\omega_{ce} - 2\omega_e\right)^2\right]\delta\hat{V}_{er}$$

$$+ \gamma_b\frac{\omega^2_{pb}}{p^2c^2}\omega^2\left(\delta\hat{V}_{br} - f_i\delta\hat{V}_{ir}\right) = -\frac{e}{m}\frac{k\omega}{p^2}\frac{d}{dr}\delta\hat{E}_z, \tag{38}$$

$$\left[\left(\eta f_i\frac{\omega^2_{pb}}{p^2c^2} + 1\right)\omega^2 - \left(\omega_{ci} + 2\omega_i\right)^2\right]\delta\hat{V}_{ir}$$

$$- \eta\frac{\omega^2_{pb}}{p^2c^2}\omega^2\left(\delta\hat{V}_{br} + f_e\delta\hat{V}_{er}\right) = \frac{e}{m_i}\frac{k\omega}{p^2}\frac{d}{dr}\delta\hat{E}_z, \tag{39}$$

for the perturbed radial mean fluid velocities $\delta\hat{V}_{jr}(r)$ which has solutions of the form

$$\nu_j^2(\omega,k)\ \delta\hat{V}_{jr} = \frac{e_j}{m_j}\frac{(\omega - kV_{jz}^0)k}{p^2}\frac{d}{dr}\delta\hat{E}_z(r). \tag{40}$$

The parameter $\nu_j^2(\omega,k)$ in Eq. (40) can be found by solving Eqs. (37) - (39) simultaneously. Note from Eq. (40) that the perturbed mean radial fluid velocity $\delta\hat{V}_{jr}$ is expressed only in terms of $(d/dr)\delta\hat{E}_z(r)$.

For the present purposes of the article which evaluates the influence of electromagnetic effects on the two stream stability behavior, we restrict present stability calculation to the long wavelength perturbations characterized by

$$kR_b << 1. \tag{41}$$

According to the traditional one dimensional two stream theory, the unstable k-values in a typical present experiment of a relativistic electron beam propagation can easily satisfy Eq. (41). In the context of Eq. (41), it is not necessary to explicitly find the parameter $\nu_j^2(\omega,k)$ in Eq. (40). From Eq. (33), the perturbed mean axial fluid velocity is also given by

$$\delta\hat{V}_{jz} = i\,\frac{e_j}{\gamma_j^2 m_j}\frac{1}{(\omega - kV_{jz}^0)}\delta\hat{E}_z. \tag{42}$$

The perturbation in density in Eq. (34) can also be eliminated in favor of $\delta\hat{E}_z$. Then, after a straightforward algebra, it is shown that the eigenvalue equation in Eq. (23) can be expressed in the form

$$\frac{1}{r}\frac{d}{dr}\left\{r\left[1 - \sum_j \left(1 - \frac{\omega V_{jz}^0}{c^2 k}\right)\frac{\omega_{pj}^2 k^2}{\nu_j^2 p^2}\frac{d}{dr}\delta\hat{E}_z\right]\right\}$$

$$- p^2\left[1 - \sum_j \frac{\omega_{pj}^2/\gamma_j^2}{(\omega - kV_{jz}^0)^2}\right]\delta\hat{E}_z = 0, \tag{43}$$

where $\omega_{pj}^2 = 4\pi e^2\hat{n}_j/m_j$ is the plasma frequency-squared for particles of component j and $p^2 = k^2 - \omega^2/c^2$ is defined in Eq. (36). Note $\omega_{pi}^2 = nf_i\omega_{pb}^2$ and $\omega_{pe}^2 = \gamma_b f_e \omega_{pb}^2$.

Inside the beam-plasma column $(0 < r < R_b)$, Eq. (43) can be expressed in the form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\delta\hat{E}_z\right) + T^2\delta\hat{E}_z = 0, \tag{44}$$

where

$$T^2 = -p^2\frac{1 - \sum_j \frac{\omega_{pj}^2/\gamma_j^2}{(\omega - kV_{jz}^0)^2}}{1 - \sum_j \left(1 - \frac{\omega V_{jz}^0}{c^2 k}\right)\frac{\omega_{pj}^2 k^2}{\nu_j^2 p^2}}. \tag{45}$$

Outside the beam-plasma column $(R_b < r < R_c)$, Eq. (43) reduces to Poisson equation in free space, that is,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\delta\hat{E}_z\right) - p^2\delta\hat{E}_z = 0, \qquad R_b < r < R_c. \tag{46}$$

Choosing a right solution which has a finite value at $r = 0$ and properly matching the boundary conditions of the beam-plasma surface $(r = R_b)$, we eventually obtain the dispersion relation[8,9]

$$\left[1 - \sum_j \left(1 - \frac{\omega V^0_{jz}}{c^2 k}\right) \frac{\omega^2_{pj} k^2}{\nu^2_j p^2}\right] TR_b \frac{J_0'(TR_b)}{J_0(TR_b)} = g(p)$$

$$= pR_b \frac{K_0(pR_c)\ I_0'(pR_b) - K_0'(pR_b)\ I_0(pR_c)}{K_0(pR_c)\ I_0(pR_b) - K_0(pR_b)\ I_0(pR_c)} \tag{47}$$

where $J_0(x)$ is the Bessel function of the first kind of order zero, and $I_0(x)$ and $K_0(x)$ are modified Bessel functions of the first and second kinds, respectively, of order zero, and the geometrical function $g(p)$ represents the right-hand side of Eq. (47). The "prime" notation in Eq. (47) denotes derivatives with respect to the complete argument of the Bessel function, e.g., $I_0'(x) = [d/dx]\ I_0(x)$. The dispersion relation in Eq. (47) is one of the main results of this article, which can be used to investigate the two stream stability properties for a broad range of physical parameters. Particularly, the influence of electromagnetic effects are incorporated into Eq. (47) through the parameter $p^2 = k^2 - \omega^2/c^2$. Although the dispersion relation in Eq. (47) is very similar to that of the electrostatic approximation,[8,9] the apparently minor modification p instead of k makes a considerable difference in determining the critical limiting beam current due to the two stream instability, as will be seen in the next section.

IV. TWO STREAM STABILITY PROPERTIES

In this section, making use of the dispersion relation in Eq. (47), we investigate two stream stability properties for long wavelength perturbations characterized by Eq. (41). The dispersion relation in Eq. (47) also represents fluctuations of the transverse field components $\delta\hat{E}_r$ and $\delta\hat{B}_\theta$, which may drive the transverse oscillation of the beam and plasma column. For example, it is more involving but straightforward to obtain the dispersion relation of non-axisymmetric electromagnetic perturbations. From this dispersion relation, we can identify typical transverse oscillations such as the ion resonance instability[10] resulted from the transverse oscillation of the beam electrons and plasma ions, and the electron-electron two-rotating stream instability[11] raised from the relative rotational difference between the beam electrons and plasma electrons which is unavoidably introduced by the virtual laminar flow assumption in the cold fluid model. The instabilities associated with the transverse oscillations are other important issues in the electron beam propagation experiments and they must be separately investigated with more attention. For the present purposes, we therefore concentrate on the two stream instability, which is a characteristic feature of fluctuations of the axial electric field. However, long wavelength stability analysis somehow eliminates interference of the transverse oscillation, thereby leaving behind dispersion relation of the two stream instability, which will be used in this section for further study. For the long wavelength perturbations satisfying Eq. (41), the left-hand side of Eq. (47) can be approximated by

$$\rho^2 R_b^2 - \rho^2 R_b^2 \sum_j \frac{\omega_{pj}^2/\gamma_j^2}{(\omega - kV_{jz}^0)^2} = 2g(\rho), \tag{48}$$

where use has been made of $|T|^2 R_b^2 < 1$. Within the context of the assumption $p^2 R_b^2 << |T|^2 R_b{}^2 < 1$, we can neglect the first term in the left-hand side of Eq. (48). However, for clarity in the subsequent stability discussion, we keep this term without effecting final results in this article. Before proceeding stability analysis of Eq. (47), we investigate general properties of the two stream dispersion function.

IV.A Analysis of Two Stream Dispersion Function

Here, we briefly investigate properties of a typical dispersion function of the two stream instability, which is expressed as

$$f(x) = \frac{a^2}{(x - \zeta)^2} + \frac{b^2}{x^2} = 1, \tag{49}$$

where a, b and ζ are positive constants, and x represents complex eigenfrequency. In the (f,x) parameter space, f(x) has its local minimum at[7,9]

$$x = x_p = \zeta \frac{(b/a)^{2/3}}{(b/a)^{2/3} + 1}. \tag{50}$$

The condition for instability is obtained from $f(x_p) > 1$ and is given by

$$\zeta < b \left[1 + (a/b)^{2/3}\right]^{3/2}. \tag{51}$$

In the limiting case when $b^2 >> a^2$, the minimum point x_p occurs at $x_p \simeq \zeta$. After a careful examination of the dispersion relation, we can also show that the maximum growth rate occurs at the axial wavelength satisfying

$$\zeta \approx b, \tag{52}$$

with its corresponding solution

$$x \approx b\left[1 \pm i\,\frac{\sqrt{3}}{2}\left(\frac{a}{2b}\right)^{2/3}\right], \tag{53}$$

for $b^2 >> a^2$. On the other hand, in the limiting case $b^2 << a^2$, the solution $x \approx (1/2)\,(1 \pm i\sqrt{3}\,)\,(ab^2/2)^{1/3}$ of Eq. (49) for the maximum growth rate occurs at the wavenumber $\zeta = a$.

IV.B Two Stream Instability in a Magnetized Beam-Plasma System

Two stream stability properties of a relativistic electron beam propagating through a magnetized plasma channel are investigated in this subsection, assuming that the beam-plasma system is bounded by an outer conductor with radius R_c. Since the plasma in the channel is strongly magnetized $(B_0 \neq 0)$, not only ions but also high density plasma electrons are allowed in the channel, still satisfying the equilibrium condition in Eq. (21). Further assuming that the axial wavelength of the perturbation is sufficiently long that

$$p^2 R_c^2 << 1, \tag{54}$$

it is straightforward to show that the right-hand side of Eq. (47) can be approximated by

$$g = -\frac{1}{\ln(R_c/R_b)}, \tag{55}$$

assuming that the conducting radius R_c is reasonably larger than the beam radius R_b. Otherwise, the approximation in Eq. (55) is failing. Therefore, the dispersion relation in Eq. (48) is approximately written by

$$\frac{2}{\ln(R_c/R_b)} + \left(k^2 - \frac{\omega^2}{c^2}\right) R_b{}^2$$

$$= \left(k^2 - \frac{\omega^2}{c^2}\right) R_b^2 \left[\frac{\omega_{pb}^2/\gamma_b^2}{(\omega - k\beta_b c)^2} + \frac{(nf_i + \gamma_b f_e)\omega_{pb}^2}{\omega^2}\right], \tag{56}$$

which can be numerically solved for the complex eigenfrequency ω.

In order to analytically track stability properties of Eq. (56), we define

$$k_\perp^2 = 2/R_b^2 \ln(R_c/R_b). \tag{57}$$

Substituting Eq. (57) into Eq. (56) and rearranging the terms, we can express Eq. (56) as

$$\frac{p^2}{k_0^2}\left[\frac{\omega_{pb}^2/\gamma_b^2}{(\omega - k\beta_b c)^2} + \frac{\gamma_b f_e \omega_{pb}^2}{\omega^2}\right] = 1, \tag{58}$$

where $k_0^2 = k_\perp^2 + p^2$, the term proportional to nf_i is simply dropped since $nf_i << \gamma_b f_e$ in a typical experiment. Identifying $a^2 = \omega_{pb}^2 p^2/\gamma_b^2 k_0^2$, $b^2 = \gamma_b f_e \omega_{pb}^2 p^2/k_0^2$, $\zeta = k\beta_b c$ and $x = \omega$, Eq. (58) is identical in structure to Eq. (49).

From Eq. (51), the axial wavenumber k of the instability borderline is approximately given by

$$(k\beta_b c)^2 = \gamma_b f_e \frac{\omega_{pb}^2}{k_0^2} p^2 \left[1 + \frac{1}{\gamma_b f_e^{1/3}}\right]^3, \tag{59}$$

in which the function f(x) [defined in Eq. (49)] has a local minimum value of unity at

$$\omega = k\beta_b c \frac{\gamma_b f_e^{1/3}}{\gamma_b f_e^{1/3} + 1} . \tag{60}$$

In obtaining Eqs. (59) and (60), we have assumed that ρ^2 is independent of the eigenfrequency ω. However, it is a function of ω in reality. Therefore, these calculations are approximations. For better understanding, Eq. (56) must be numerically solved. Substituting Eq. (60) into Eq. (36), it is straightforward to show

$$p^2 = k^2/\xi_{th}, \tag{61}$$

at the axial wavenumber k satisfying the stability boundary in Eq. (59). In Eq. (61), the electromagnetic enhancement factor ξ_{th} is defined by

$$\xi_{th} = \frac{(\gamma_b f_e^{1/3} + 1)^2}{2\gamma_b f_e^{1/3} + f_e^{2/3} + 1}. \tag{62}$$

Strictly speaking, the theoretically obtained enhancement factor ξ_{th} in Eq. (62) is an approximation. The meaning of "enhancement" will be apparent in the following discussion. However, we emphasize that in the electrostatic approximation, the factor ξ_{th} in Eq. (62) is unity for arbitrary value of

γ_b. Any value of the enhancement factor ξ_{th} which is larger than unity is the outcome of electromagnetic effects.

Replacing ρ^2 in Eq. (59) by k^2/ξ_{th} and rearranging terms, we have the relation

$$(\kappa\beta_b c)^2 = \gamma_b f_e \omega_{pb}^2 \left[1 + \frac{1}{\gamma_b f_e^{1/3}}\right]^3 - k_\perp^2 \xi_{th} \beta_b^2 c^2, \tag{63}$$

for the stability criterion. In obtaining Eq. (63), use has been made of $k_0^2 = \rho^2 + k_\perp^2$. It is apparent from Eq. (63) that the electromagnetic influence represented by ξ_{th} increases the effective transverse wavenumber from $k_\perp$ to $k_\perp \xi_{th}^{1/2}$. In other words, rewriting the second term in the right-hand side of Eq. (63) by

$$k_\perp^2 \xi_{th} \beta_b^2 c^2 = \frac{2\beta_b^2 c^2/\ln\left(R_c/R_b\right)}{R_b^2/\xi_{th}},$$

we observe that the effective beam radius reduces from R_b to $R_b/\xi_{th}^{1/2}$, thereby enhancing the stabilizing influence[8,9] of finite radial geometry effects on stability behavior. It is also evident from Eq. (63) that for instability, the beam plasma frequency ω_{pb} must satisfy

$$\omega_{pb}^2 > \frac{2\xi_{th}\beta_b^2 c^2/R_b^2 \ln(R_c/R_b)}{\gamma_b f_e\left(1 + 1/\gamma_b f_e^{1/3}\right)^3}, \tag{64}$$

which is necessary condition for instability. For fixed values of $\hat{n}_b$, γ_b and f_e, the instability condition in Eq. (64) cannot be satisfied if R_b is sufficiently small, that is, finite radial geometry effects have a stabilizing influence on the two stream instability.

In terms of the magnitude of the beam current $I_b = \hat{n}_b e\beta_b c\pi R_b^2$, it is straightforward to show that the instability condition in Eq. (64) can be expressed in the equivalent form,

$$I_b > I_{crit}, \tag{65}$$

where the critical current I_{crit} for instability is given by

$$I_{crit} = \frac{I_A}{2\ln(R_c/R_b)} \frac{\xi_{th}\beta_b^3/f_e}{\left(1 + 1/\gamma_b f_e^{1/3}\right)^3}. \tag{66}$$

In Eq. (66) $I_A = mc^3/e \approx 17000$ amperes is the Alfven critical current. The electron beam with current below the critical current is stable whereas the beam with current above the critical current is unstable. The critical current in Eq. (66) is proportional to the current enhancement factor ξ_{th} which is defined in Eq. (62). In the limiting case when $\gamma_b = 1$, the enhancement factor ξ_{th} in Eq. (62) recovers the electrostatic approximation $\xi_{th} = 1$ for arbitrary value of f_e. In order to dramatically demonstrate dependence of ξ_{th} on the beam energy γ_b, we simplify Eq. (62) by

$$\xi_{th} = \frac{\gamma_b + 1}{2}, \tag{67}$$

for the equidensity case characterized by $f_e = 1$. Obviously from Eqs. (66) and (67), the critical beam current for instability increases drastically with the beam energy γ_b.

In order to complete the stability analysis in this subsection, we numerically solve the dispersion relation in Eq. (56). For references in the

subsequent stability analysis, we define normalized values of the critical beam plasma frequency-squared

$$S_e = \left(\frac{\omega_{pb}^2 R_b^2}{c^2}\right)_{crit} = \frac{2\beta_b^2/\ln(R_c/R_b)}{\gamma_b f_e(1 + 1/\gamma_b f_e^{1/3})^3}, \tag{68}$$

for the electrostatic approximation,

$$S_m = \frac{2\xi_{th}\beta_b^2/\ln(R_c/R_b)}{\gamma_b f_e(1 + 1/\gamma_b f_e^{1/3})^3}, \tag{69}$$

from Eq. (64) for the approximated electromagnetic calculation. Note that an electron beam with plasma frequency below the critical plasma frequency is stable whereas the beam with frequency above the critical plasma frequency is unstable. We also remind the reader that the critical frequency in Eq. (69) is also an approximation. From a numerical calculation of Eq. (56), we have obtained correct value of normalized critical plasma frequency - squared S which represents the stability boundary in $(\omega_{pb}^2 R_b^2/c^2, \gamma_b)$ parameter space. Shown in Fig. 1 is plot of S versus γ_b obtained from Eq. (56) for $R_c/R_b = 7.4$ and $f_e = 1$. As references, we also present S_e and S_m in Fig. 1.

The current enhancement factor ξ_{th} in Eq. (62) has been derived from the approximation in Eq. (60). Note $\xi_{th} = S_m/S_e$ from Eqs. (68) and (69). Similarly, we numerically determine the electromagnetic current enhancement factor ξ from

$$\xi = S/S_e, \tag{70}$$

where no approximation has been made. Figure 2 presents plot of ξ versus γ_b obtained from Eq. (56) for parameters identical to Fig. 1. The theoretically obtained ξ_{th} is also presented in Fig. 2 as a reference. Obviously from Fig. 2, we note that the theoretical enhancement factor ξ_{th} in Eq. (62) is underestimating the true electromagnetic current enhancement. In this regard, the true critical current I_{crit} for instability is given by Eq. (66) where the theoretical value of ξ_{th} is replaced by the numerical evaluation of ξ. As a comparison with Fig. 2, we also present plot of ξ versus γ_b in Fig. 3 for $f_e = 10$ and $R_c/R_b = 7.4$ where the plasma electron density is ten times higher than that in Fig. 2. As expected from Eq. (62), the electromagnetic current enhancement increases with the plasma density.

IV.C Two Stream Stability Properties in a Self-Pinched Electron Beam

As a second example, we investigate two stream stability properties in a self-pinched relativistic electron beam propagating through a pure ion channel. Because of $B_0 = 0$, there is no externally provided radial confinement force for plasma electrons, thereby not allowing the plasma electrons in the channel. Moreover, the ion density in the channel must satisfy $f_i < 1$ in order for a radially confined equilibrium for ions [see Eqs. (18) and (19)]. We, therefore, concentrate on the two stream analysis resulted from the relative drift motion between the beam electrons and channel ions. In addition, we also assume that the beam channel system is unbounded with $R_c \to \infty$. Taylor expanding the right-hand side of Eq. (47) for long wavelength perturbations [Eqs. (41)] and making use of Eq. (48), the dispersion relation can be expressed in the approximate form

$$\frac{2}{R_b^2 \ln(1/pR_b)} + p^2 \left[1 - \frac{\omega_{pb}^2/\gamma_b^2}{(\omega - k\beta_b c)^2} - \frac{nf_i\omega_{pb}^2}{\omega^2}\right] = 0, \tag{71}$$

where p^2 is defined in Eq. (36).

Numerical calculation of the dispersion relation in Eq. (71) is similar to that of Eq. (56). Therefore, instead of carrying out a numerical exercise of Eq. (71), we briefly investigate properties of Eq. (71) analytically, although the analytical approximation underestimate the influence of electromagnetic effects on stability behavior as shown in the previous subsection. Making use of the approximation of p^2 in Eq. (61), Eq. (71) is simplified to

$$\frac{k^2}{k^2 + k_\perp^2} \left[\frac{\omega_{pb}^2/\gamma_b^2}{(\omega - k\beta_b c)^2} + \frac{nf_i\omega_{pb}^2}{\omega^2}\right] = 1, \tag{72}$$

where $k_\perp^2$ is defined by

$$k_\perp^2 = \frac{2\varepsilon_{th}}{R_b^2 \left[\ln(1/kR_b) + \frac{1}{2}\ln\varepsilon_{th}\right]}. \tag{73}$$

Obviously, accurate instability boundaries for the axial wavelength must be numerically found from Eq. (72). However, for the present purposes, we make the assumption that if the perturbations of the axial wave number

$$kR_b = 10^{-9}, \tag{74}$$

are stable, the beam ion two stream instability is stable in a practical sense.[9] Substituting Eq. (74) into Eq. (73) and paralleling the derivation of Eq. (66), we obtain the critical beam current

$$I_{crit} = \frac{0.425\ \xi_{th}\gamma_b^3\beta_b^3}{[1 + (\ln\xi_{th})/40][1 + (nf_i\gamma_b^2)^{1/3}]^3}\ (kA), \qquad (75)$$

for instability. Here, the critical current has been expressed in units of kiloampere. In obtaining Eq. (75), use has been made of the approximation $9 \ln 10 \approx 20$. Note from Eq. (71) that the critical current I_{crit} is proportional to the enhancement factor ξ_{th}. Once again we therefore conclude that the electromagnetic effects strongly enhance the critical beam current for the two stream instability resulted from the relative motion between the beam electrons and ions.

V. CONCLUSIONS

In this paper, we have investigated the influence of electromagnetic effects on the two stream instability in a relativistic electron beam propagating through a collisionless plasma channel. The equilibrium and stability analysis (Secs. II-IV) was carried out within the framework of a macroscopic cold fluid model in which the beam and plasma fluid element is in a laminar flow. Moreover, axisymmetric electromagnetic stability properties were calculated for the case in which the equilibrium beam and plasma density profiles are rectangular. Consistent with the two stream instability, the perturbations are polarized with the transverse magnetic modes. A general dispersion relation of the transverse magnetic modes has been obtained in Sec. III. This dispersion relation has been applied to two specific cases. One of the striking features of the stability analysis is that the critical beam current for instability is proportional to the electromagnetic current enhancement factor ξ, which increases drastically with the beam energy (γ_b). For example, the equidensity case, the current enhancement factor ξ is approximately given by $\xi = (\gamma_b + 1)/2$ for the beam electron - plasma electron two stream instability. Thus, for a relativistic electron beam with $\gamma_b >> 1$, the critical beam current obtained from the electromagnetic calculation can be several times larger than that from the electrostatic approximation. We, therefore, conclude that the influence of electromagnetic effects on the two stream stability behavior plays a major role in determining various physical parameters for a long distance propagation of relativistic electron beams.

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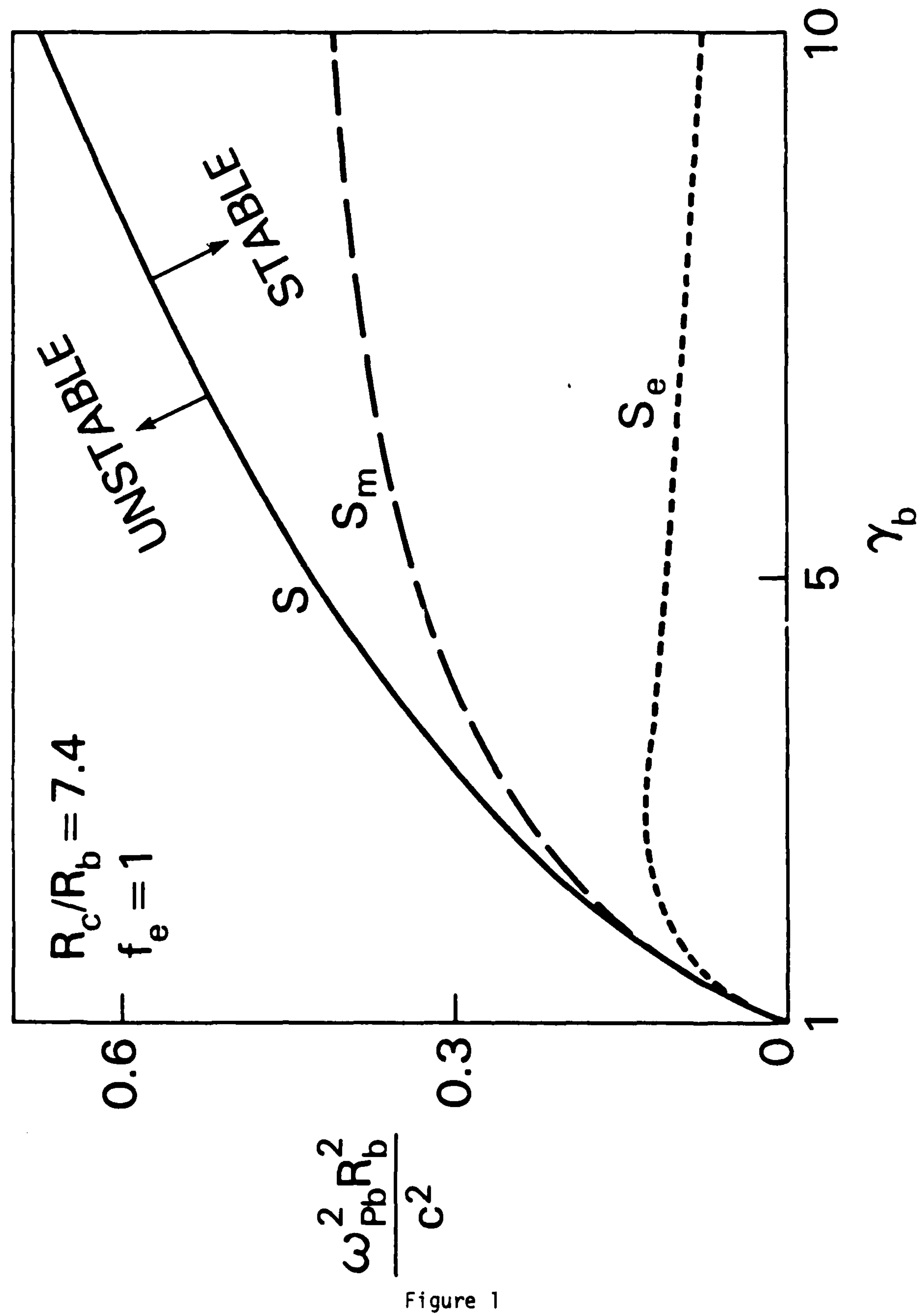
FIGURE CAPTIONS

Figure 1 Stability boundary in $(\omega_{pb}^2 R_b^2/c^2, \gamma_b)$ parameter space for $R_c/R_b = 7.4$ and $f_e = 1$. S is numerically obtained from Eq. (56). S_e and S_m are approximations obtained from Eqs. (68) and (69), respectively.

Figure 2 Plot of ξ versus γ_b obtained from Eq. (56) for the parameters identical to Figure 1. As a reference, ξ_{th} is also presented.

Figure 3 Plot of ξ versus γ_b obtained from Eq. (56) for R_c/R_b = 7.4 and $f_e = 10$.



$R_c/R_b = 7.4$
$f_e = 1$
UNSTABLE
STABLE
S
S_m
S_e
0.6
0.3
0
1
5
10
γ_b
$\frac{\omega_{Pb}^2 R_b^2}{c^2}$

Figure 1

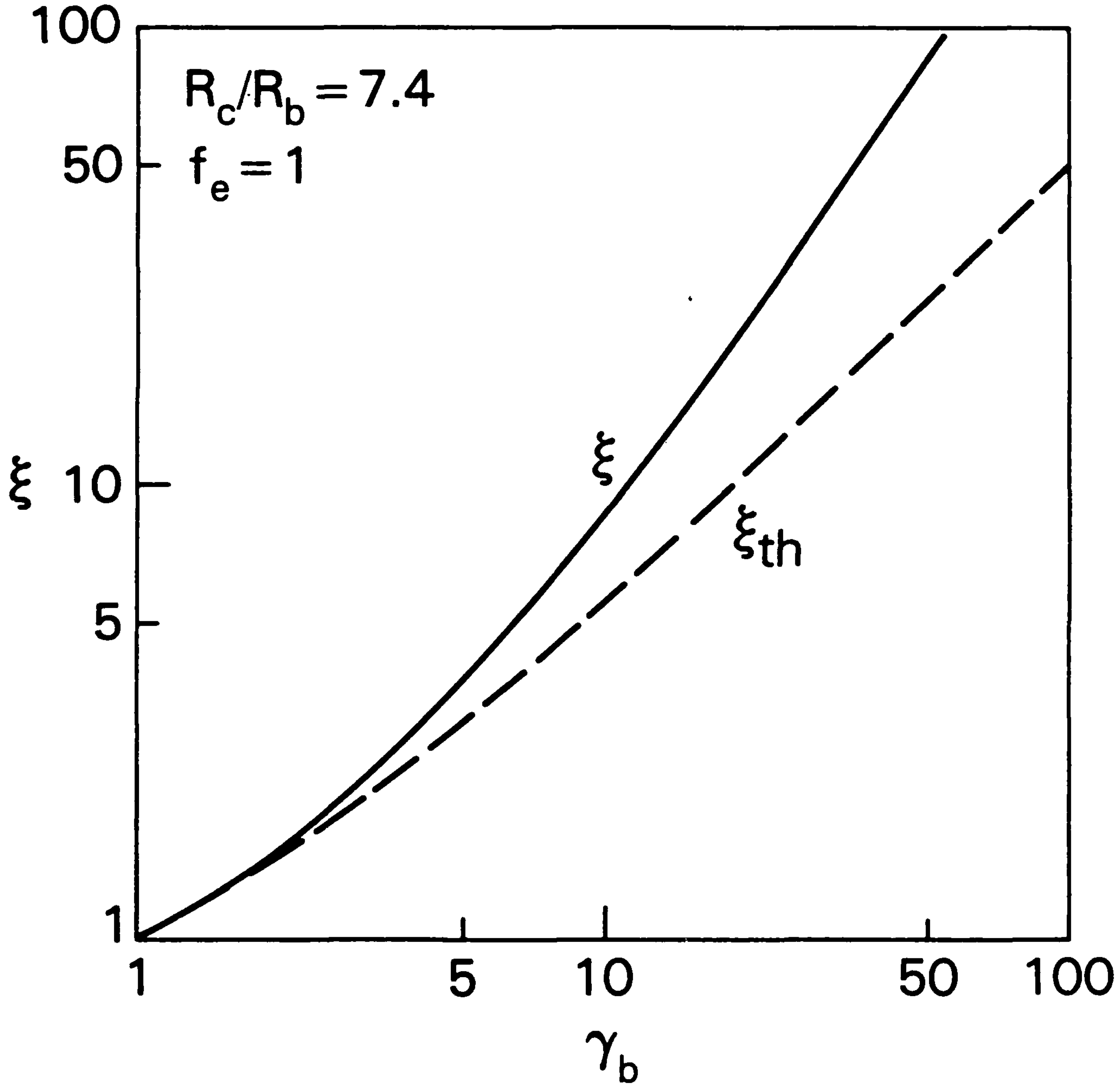


Figure 2

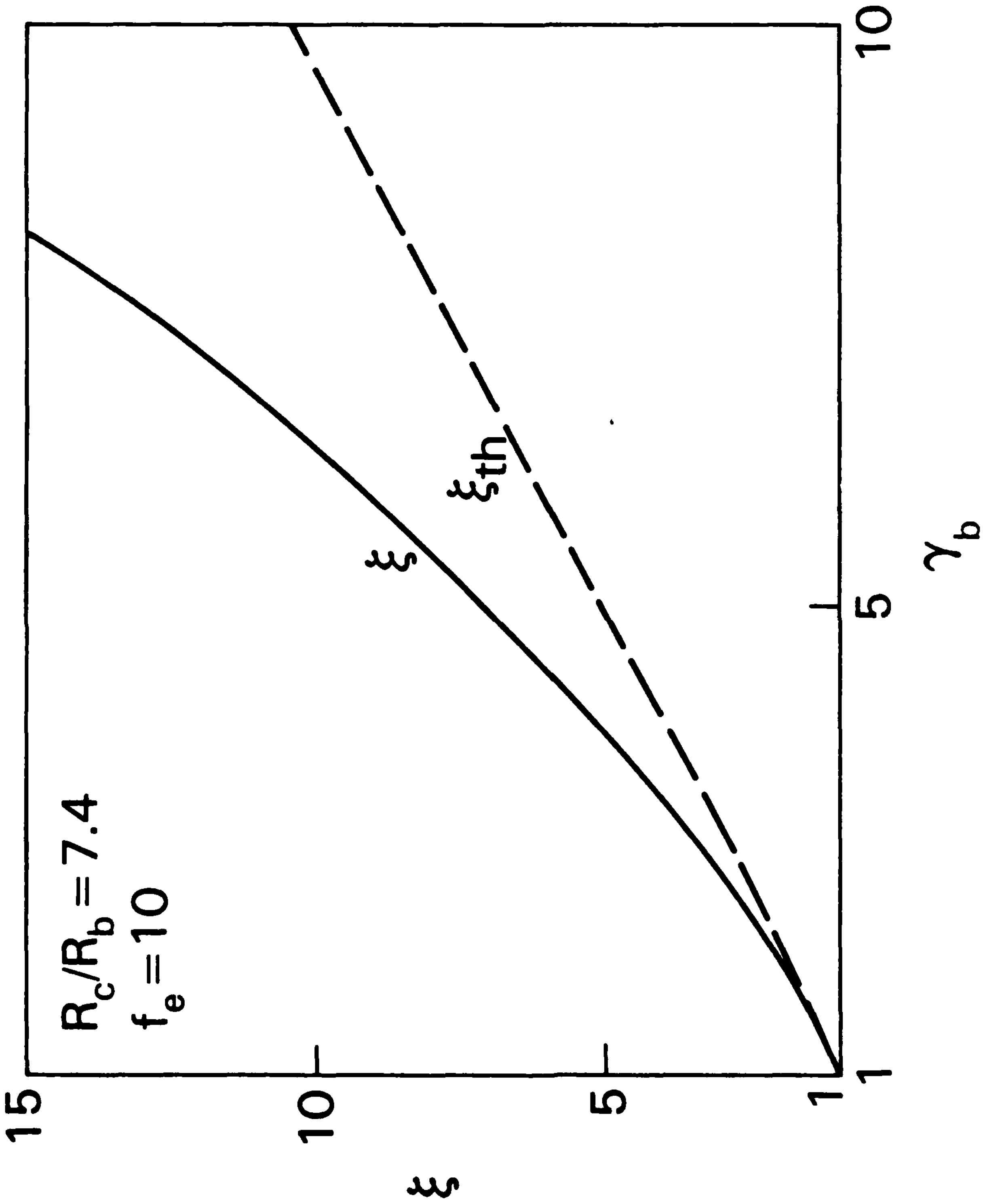


Figure 3

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